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Abstract: In this paper we describe the Seiberg-Witten invariants, which have been introduced by Witten, for manifolds with $b_+ = 1$. In this case the invariants depend on a chamber structure, and there exists a universal wall crossing formula. For every Kähler surface with $p_g = 0$ and q=0, these invariants are non-trivial for all $Spin^c(4)$-structures of non-negative index.

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Seiberg-Witten invariants for manifolds with $b_+ = 1$, and the universal wall crossing formula

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1 Introduction

The purpose of this paper is to give a systematic description of the Seiberg-Witten invariants, which were introduced in [W], for manifolds with $b_+ = 1$. In this situation, compared to the general case $b_+ > 1$, several new features arise.

The Seiberg-Witten invariants for manifolds with $b_+ > 1$ are (non - homogeneous) forms $SW_{X,\mathcal{O}}(\mathfrak{c}) \in \Lambda^* H^1(X, \mathbb{Z})$, associated with an orientation parameter $\mathcal{O}$ and a class of a $Spin^c(4)$-structures $\mathfrak{c}$ on X.

The invariants for manifolds with $b_+ = 1$ depend on a chamber structure; they are associated with data $(\mathcal{O}_1, \mathbf{H}_0, \mathfrak{c})$, where $(\mathcal{O}_1, \mathbf{H}_0)$ are orientation parameters and $\mathfrak{c}$ is again the class of a $Spin^c(4)$-structure on X. In this case, the invariants are functions $SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c}) : \{\pm\} \longrightarrow \Lambda^* H^1(X, \mathbb{Z})$.

One of the main results of this paper is the proof of a universal wall crossing formula. This formula, which generalizes previous results of [W], [KM] and [LL] describes the difference $SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})(+) - SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})(-)$ as an

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abelian $Spin^c(4)$-form. More precisely, on elements $\lambda \in \Lambda^r \left({H_1(X,\mathbb{Z})}/{\mathrm{Tors}} \right)$ with $0 \leq r \leq \min(b_1, w_{\mathfrak{c}})$, we have:

$$\left[SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})(+) - SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})(-) \right](\lambda) = \langle \lambda \wedge \exp(-u_c), l_{\mathcal{O}_1} \rangle \ ,$$

where $u_c \in \Lambda^2 \left({H_1(X,\mathbb{Z})}/{\mathrm{Tors}} \right)$ is given by $u_c(a \wedge b) = \frac{1}{2}\langle a \cup b \cup c, [X] \rangle$, $a, b \in H^1(X,\mathbb{Z})$, and $l_{\mathcal{O}_1} \in \Lambda^{b_1} H^1(X,\mathbb{Z})$ represents the orientation $\mathcal{O}_1$ of $H^1(X,\mathbb{R})$. Here c is the Chern class of $\mathfrak{c}$ and $w_{\mathfrak{c}} := \frac{1}{4}(c^2 - 3\sigma(X) - 2e(X))$ is the index of $\mathfrak{c}$.

This formula has some important consequences, e.g. it shows that Seiberg-Witten invariants of manifolds with positive scalar curvature metrics are essentially topological invariants. According to Witten's vanishing theorem [W], one has $SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})(\pm) = 0$ for at least one element of $\{\pm\}$, and the other value is determined be the wall crossing formula.

In the final part of the paper we show how to calculate $SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})(\pm)$ for Kählerian surfaces. The relevant Seiberg-Witten moduli spaces have in this case a purely complex analytic description as Douady spaces of curves representing given homology classes: this description is essentially the Kobayashi-Hitchin correspondence obtained in [OT1].

Witten has shown that non-trivial invariants of Kählerian surfaces with $b_+ > 1$ must necessarily have index 0. This is not the case for surfaces with $b_+ = 1$. We show that a Kählerian surface with $b_+ = 1$ and $b_1 = 0$ has $SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})(\{\pm\}) = \{0, 1\}$ or $SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})(\{\pm\}) = \{0, -1\}$ as soon as the index of $\mathfrak{c}$ is non-negative. For these surfaces the invariants are therefore completely determined by their reductions modulo 2.

There exist examples of 4-manifolds with $b_+ = 1$ which possess - for every prescribed non-negative index - infinitely many classes $\mathfrak{c}$ of $Spin^c(4)$-structures with $SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c}) \not\equiv 0$.

2 The twisted Seiberg-Witten equations

Let X be a closed connected oriented 4-manifold, and let $c \in H^2(X,\mathbb{Z})$ be a class with $c \equiv w_2(X)$ (mod 2). A compatible $Spin^c(4)$-bundle is a $Spin^c(4)$-bundle $\hat{P}$ over X with $c_1(\hat{P} \times_{\det} \mathbb{C}) = c$ such that its $GL_+(4,\mathbb{R})$-extension $\hat{P} \times_{\tilde{\pi}} GL_+(4,\mathbb{R})$ is isomorphic to the bundle of oriented frames in Λ^1_X; here

$\tilde{\pi}$ denotes the composition of the canonical representation $\pi : Spin^c(4) \longrightarrow SO(4)$ with the inclusion $SO(4) \subset GL_+(4, \mathbb{R})$. Let $\Sigma^\pm := \hat{P} \times_{\sigma_\pm} \mathbb{C}^2$ be the associated spinor bundles with $\det \Sigma^\pm = \hat{P} \times_{\det} \mathbb{C}$ [OT1].

Definition 1 *A* <u>*Clifford*</u> <u>*map*</u> *of type $\hat{P}$ is an orientation-preserving isomorphism* $\gamma : \Lambda^1_X \longrightarrow \hat{P} \times_\pi \mathbb{R}^4$.

The $SO(4)$-vector bundle $\hat{P} \times_\pi \mathbb{R}^4$ can be identified with the bundle $\mathbb{R}SU(\Sigma^+, \Sigma^-)$ of real multiples of $\mathbb{C}$-linear isometries of determinant 1 from Σ^+ to Σ^-.

A Clifford map γ defines a metric g_γ on X, a lift $\hat{P} \longrightarrow P_{g_\gamma}$ of the associated frame bundle, and it induces isomorphisms $\Gamma : \Lambda^2_\pm \longrightarrow su(\Sigma^\pm)$ [OT1]. We denote by $\mathcal{C} = \mathcal{C}(\hat{P})$ the space of all Clifford maps of type $\hat{P}$. The quotient ${}^{\mathcal{C}}\!/_{\mathrm{im}[\mathrm{Aut}(\hat{P}) \longrightarrow \mathrm{Aut}(\hat{P} \times_\pi SO(4))]}$ parametrizes the set of all $Spin^c(4)$-structures of Chern class c, whereas ${}^{\mathcal{C}}\!/_{\mathrm{Aut}(\hat{P} \times_\pi SO(4))}$ can be identified with the space $\mathcal{M}et_X$ of Riemannian metrics on X. In fact, since $\mathcal{M}et_X$ is contractible, we have a natural isomorphism ${}^{\mathcal{C}}\!/_{\mathrm{Aut}(\hat{P})} \xrightarrow{\simeq} \mathcal{M}et_X \times \pi_0\left({}^{\mathcal{C}}\!/_{\mathrm{Aut}(\hat{P})}\right)$, where the second factor is a $\mathrm{Tors}_2 H^2(X, \mathbb{Z})$-torsor; it parametrizes the set of equivalence classes of $Spin^c(4)$ -structures with Chern class c on (X, g), for an arbitrary metric g. The latter assertion follows from the fact that the map $H^1(X, \mathbb{Z}_2) \longrightarrow H^1(X, \underline{Spin^c}(4))$ is trivial for any 4-manifold X. We use the symbol $\mathfrak{c}$ to denote elements in $\pi_0\left({}^{\mathcal{C}}\!/_{\mathrm{Aut}(\hat{P})}\right)$, and we denote by $\mathfrak{c}_\gamma$ the connected component defined by $[\gamma] \in {}^{\mathcal{C}}\!/_{\mathrm{Aut}(\hat{P})}$.

A fixed Clifford map γ defines a bijection between unitary connections in $\hat{P} \times_{\det} \mathbb{C}$ and $Spin^c(4)$-connections in $\hat{P}$ which lift (via γ) the Levi-Civita connection in P_{g_γ}, and allows to associate a Dirac operator $\not{D}_A$ to a connection $A \in \mathcal{A}(\hat{P} \times_{\det} \mathbb{C})$.

Definition 2 *Let γ be a Clifford map, and let $\beta \in Z^2_{\mathrm{DR}}(X)$ be a closed 2-form. The β-twisted Seiberg-Witten equations are*

$$\left\{ \begin{array}{lll} \not{D}_A \Psi & = & 0 \\ \Gamma\left((F_A + 2\pi i \beta)^+\right) & = & 2(\Psi\bar{\Psi})_0 \ . \end{array} \right. \qquad (SW^\gamma_\beta)$$

These twisted Seiberg-Witten equations arise naturally in connection with certain non-abelian monopoles [OT2], [T]. They should not be regarded as perturbation of (SW_0^γ), since later the cohomology class of β will be fixed.

Let $\mathcal{W}^\gamma_{X,\beta}$ be the moduli space of solutions $(A,\Psi)\in\mathcal{A}(\det\Sigma^+)\times A^0(\Sigma^+)$ of (SW^γ_β) modulo the natural action $((A,\Psi),f)\longmapsto(A^{f^2},f^{-1}\Psi)$ of the gauge group $\mathcal{G}=\mathcal{C}^\infty(X,S^1)$.

Since two Clifford maps lifting the same pair $(g,\mathfrak{c})$ are equivalent modulo $\mathrm{Aut}(\hat{P})$, the moduli space $\mathcal{W}^\gamma_{X,\beta}$ depends up to canonical isomorphism only on $(g_\gamma,\mathfrak{c}_\gamma)$ and β.

Now fix a class $b\in H^2_{\mathrm{DR}}(X)$, consider (SW^γ_β) as equation for a triple $(A,\Psi,\beta)\in\mathcal{A}(\det\Sigma^+)\times A^0(\Sigma^+)\times b$, and let $\mathcal{W}^\gamma_{X,b}\subset\mathcal{A}(\det\Sigma^+)\times A^0(\Sigma^+)\times b\big/_{\mathcal{G}}$ be the (infinite dimensional) moduli space of solutions. Finally we need the universal moduli space $\mathcal{W}_X\subset\mathcal{A}(\det\Sigma^+)\times A^0\ (\Sigma^+)\times Z^2_{DR}(X)\times\mathcal{C}\big/_{\mathcal{G}}$ of solutions of (SW^γ_β) regarded as equations for tuples $(A,\Psi,\beta,\gamma)\in\mathcal{A}(\det\Sigma^+)\times A^0(\Sigma^+)\times Z^2_{DR}(X)\times\mathcal{C}$.

We complete the spaces $\mathcal{A}(\det\Sigma^+)$, $A^0(\Sigma^\pm)$ and A^2 with respect to the Sobolev norms L^2_q, L^2_q and L^2_{q-1}, and the gauge group $\mathcal{G}$ with respect to L^2_{q+1}, but we suppress the Sobolev subscripts in our notations. As usual we denote by the superscript * the open subspace of a moduli space where the spinor component is non-zero.

Definition 3 *Let $c\in H^2(X,\mathbb{Z})$ be characteristic. A pair $(g,b)\in\mathcal{M}et_X\times H^2_{\mathrm{DR}}(X)$ is c-good if the g-harmonic representant of $(c-b)$ is not g-anti-selfdual.*

A pair (g,b) is c-good for every metric g if $(c-b)\neq 0$ and $(c-b)^2\geq 0$.

Proposition 4 *Let X be a closed oriented 4-manifold, and let $c\in H^2(X,\mathbb{Z})$ be characteristic. Choose a compatible $Spin^c(4)$-bundle $\hat{P}$ and an element $\mathfrak{c}\in\pi_0\left(\mathcal{C}\big/_{\mathrm{Aut}(\hat{P})}\right)$.*
i) The projections $p:\mathcal{W}_X\longrightarrow Z^2_{DR}(X)\times\mathcal{C}$ and $p_{\gamma,b}:\mathcal{W}^\gamma_{X,b}\longrightarrow b$ are proper for all choices of γ and b.
ii) $\mathcal{W}^*_X$ and ${\mathcal{W}^\gamma_{X,b}}^*$ are smooth manifolds for all γ and b .
iii) ${\mathcal{W}^\gamma_{X,b}}^*=\mathcal{W}^\gamma_{X,b}$ if (g_γ,b) is c-good.
iv) If (g_γ,b) is c-good, then every pair (β_0,β_1) of regular values of $p_{\gamma,b}$ can be joined by a smooth path $\beta:[0,1]\longrightarrow b$ such that the fiber product $[0,1]\times_{(\beta,p_{\gamma,b})}$

$\mathcal{W}^{\gamma}_{X,b}$ defines a smooth bordism between $\mathcal{W}^{\gamma}_{X,\beta_0}$ and $\mathcal{W}^{\gamma}_{X,\beta_1}$.
v) If (g_0, b_0), (g_1, b_1) are c-good pairs which can be joined by a smooth path of c-good pairs, then there is a smooth path $(\beta, \gamma) : [0,1] \longrightarrow Z^2_{\mathrm{DR}}(X) \times \mathcal{C}$ with the following properties:

1. *$[\beta_i] = b_i$ and $g_{\gamma_i} = g_i$ for $i = 0, 1$.*
2. *γ_t lifts $(g_{\gamma_t}, \mathfrak{c})$ and $(g_{\gamma_t}, [\beta_t])$ is c-good for every $t \in [0,1]$.*
3. *$[0,1] \times_{((\beta,\gamma),p)} \mathcal{W}^*_X$ is a smooth bordism between $\mathcal{W}^{\gamma_0}_{X,\beta_0}$ and $\mathcal{W}^{\gamma_1}_{X,\beta_1}$.*

vi) If $b_+ > 1$, then any two c-good pairs (g_0, b_0), (g_1, b_1) can be joined by a smooth path of c-good pairs .

Proof:
i) See [KM], Corollary 3.
ii) It suffices to show that, for a fixed class $b \in H^2_{\mathrm{DR}}(X)$, the map

$$F : \mathcal{A}(\det \Sigma^+) \times A^0(\Sigma^+) \times b \longrightarrow A^0(\Sigma^-) \times iA^0(su(\Sigma^+))$$

given by $F(A, \Psi, \beta) = \left(\not{D}_A \Psi, \Gamma\left((F_A + 2\pi i\beta)^+\right) - 2(\Psi\bar{\Psi})_0\right)$ is a submersion in every point $\tau = (A, \Psi, \beta)$ with $\Psi \neq 0$ and $\not{D}_A\Psi = 0$. To see this write $F = (F^1, F^2)$ for the components of F, and consider a pair $(\Psi^-, S) \in A^0(\Sigma^-) \times iA^0(su(\Sigma^+)$ which is L^2-orthogonal to the image of $d_\tau(F)$. Using variations of β by exact forms, we see that $\Gamma^{-1}(S) \in iA^2_+$ is orthogonal to $id^+(A^1)$, hence must be a harmonic selfdual form. This implies $\langle d_\tau(F^2)(a,0,0), S\rangle = 0$ for any variation $a \in iA^1$, and therefore

$$Re\langle\gamma(a)(\Psi), \Psi^-\rangle = Re\left\langle d_\tau(F^1)(a,0), (\Psi^-)\right\rangle = \left\langle d_\tau(F)(a,0,0), (\Psi^-, S)\right\rangle = 0$$

for all $a \in iA^1$. Since Ψ is Dirac-harmonic and non-trivial, it cannot vanish on non-empty open sets. Therefore, the multiplication map $\gamma(\cdot)\Psi : iA^1 \longrightarrow A^0(\Sigma^-)$ has L^2-dense image, so that we must have $\Psi^- = 0$. Using now the vanishing of $\Psi^- = 0$, we get

$$\langle d_\tau F^2(0,\psi), S\rangle = \langle d_\tau F(0,\psi,0), (\Psi^-, S)\rangle = 0$$

for all variations $\psi \in A^0(\Sigma^+)$. Since $d_\tau F^2(0,\cdot) : A^0(\Sigma^+) \longrightarrow iA^0(su(\Sigma^+))$ has L^2-dense image, we must also have $S = 0$.
iii) This is clear since the form $F_A + 2\pi i\beta$ represents the cohomology class $-2\pi i(c - b)$: If $(A, 0)$ was a solution of (SW^{γ}_{β}), then the g_γ-anti-selfdual form $\frac{i}{2\pi}F_A - \beta$ would be the g_γ-harmonic representant of $c - b$.

iv) This follows from *ii)*, *iii)* and Proposition (4.3.10) of [DK].
v) This is a consequence of *ii)*, *iii)*, Proposition (4.3.10) of [DK], and the fact that the condition "c-good" is open for fixed c.
vi) For fixed c, the closed subspace of $Z^2_{\mathrm{DR}}(X) \times \mathcal{C}$ consisting of pairs (β, γ) with $(g_\gamma, [\beta])$ not c-good has codimension b_+. Its complement is therefore connected when $b_+ \geq 2$. This fact was noticed by C. Taubes [Ta]. ∎

Remark: Suppose (g, b) is a c-good pair. Then the bordism type of a smooth moduli space $\mathcal{W}^\gamma_{X,\beta}$ with $g_\gamma = g$ depends only on the pair $(g, \mathfrak{c})$ and the class b of β. Furthermore, this bordism type does not change as long as one varies (g, b) in a smooth 1-parameter family of c-good pairs. Note that the statement makes sense since the set $\pi_0({}^{\mathcal{C}}/_{\mathrm{Aut}(\hat{P})})$ to which $\mathfrak{c}$ belongs was defined independently of the metric.

3 Seiberg-Witten invariants for 4-manifolds with $b_+ = 1$

Let X be a closed connected oriented 4-manifold , c a characteristic element, and $\hat{P}$ a compatible $Spin^c(4)$-bundle. We put

$$\mathcal{B}(c)^* := {}^{\mathcal{A}(\det \Sigma^+) \times (A^0(\Sigma^+) \setminus \{0\})}/_{\mathcal{G}} \ .$$

Lemma 5 *$\mathcal{B}(c)^*$ has the weak homotopy type of $K(\mathbb{Z}, 2) \times K(H^1(X, \mathbb{Z}), 1)$. There is a natural isomorphism*

$$\nu : \mathbb{Z}[u] \otimes \Lambda^*({}^{H_1(X,\mathbb{Z})}/_{\mathrm{Tors}}) \longrightarrow H^*(\mathcal{B}(c)^*, \mathbb{Z}).$$

Proof: The inclusion $S^1 \subset \mathcal{G}$ defines a canonical exact sequence

$$1 \longrightarrow S^1 \longrightarrow \mathcal{G} \longrightarrow \overline{\mathcal{G}} \longrightarrow 1$$

with $\overline{\mathcal{G}} := {}^{\mathcal{G}}/_{S^1}$, and the exponential map yields a natural identification of $\overline{\mathcal{G}}$ with the product ${}^{\mathcal{C}^\infty(X,\mathbb{R})}/_{\mathbb{R}} \times H^1(X, \mathbb{Z})$. The choice of a base point

$x_0 \in X$ induces a splitting $ev_{x_0} : \mathcal{G} \longrightarrow S^1$ of the exact sequence, and therefore a homotopy equivalence of classifying spaces $B\mathcal{G} \longrightarrow BS^1 \times B\overline{\mathcal{G}}$; the homotopy class of this map is independent of x_0 when X is connected. Since $\mathcal{A}(\det \Sigma^+) \times (A^0(\Sigma^+) \setminus \{0\})$ is weakly contractible, $\mathcal{B}(c)^*$ has the weak homotopy type of $B\mathcal{G}$. We fix weak homotopy equivalences $\mathcal{B}(c)^* \simeq B\mathcal{G}$, $BS^1 \simeq K(\mathbb{Z}, 2)$ and $B\overline{\mathcal{G}} \simeq K(H^1(X, \mathbb{Z}), 1)$ in the natural homotopy classes. Since the homotopy class of the induced weak homotopy equivalence $\mathcal{B}(c)^* \longrightarrow K(\mathbb{Z}, 2) \times K(H^1(X, \mathbb{Z}), 1)$ is canonical, we obtain a natural isomorphism

$$H^*(\mathcal{B}(c)^*, \mathbb{Z}) \simeq H^*(K(\mathbb{Z}, 2)) \otimes H^*(K(H^1(X, \mathbb{Z}), 1)) \simeq \mathbb{Z}[u] \otimes \Lambda^*({H_1(X, \mathbb{Z})}/_{\mathrm{Tors}}) \, .$$

$\blacksquare$

Remark: Let $\mathcal{G}_0$ be the kernel of the evaluation map $ev_{x_0} : \mathcal{G} \longrightarrow S^1$, and set $\mathcal{B}_0(c)^* := {\mathcal{A}(\det \Sigma^+) \times (A^0(\Sigma^+) \setminus \{0\})}/_{\mathcal{G}_0}$. The group ${\mathcal{G}}/_{\mathcal{G}_0} \simeq S^1$ acts freely on $\mathcal{B}_0(c)^*$ and defines a principal S^1-bundle over $\mathcal{B}(c)^*$. The first Chern class of this bundle is the class u defined above.

Suppose now that (g, b) is a c-good pair, and fix $\mathfrak{c} \in \pi_0({\mathcal{C}}/_{\mathrm{Aut}(\hat{P})})$. The moduli space $\mathcal{W}^{\gamma}_{X,\beta}$ is a compact manifold of dimension $w_c := \frac{1}{4}(c^2 - 2e(X) - 3\sigma(X))$ for every lift γ of $(g, \mathfrak{c})$ and every regular value β of $p_{\gamma,b}$. It can be oriented by using the canonical complex orientation of the line bundle $\det_{\mathbb{R}}(\mathrm{index}(\not{D}))$ over $\mathcal{B}(c)^*$ together with a chosen orientation $\mathcal{O}$ of the line $\det(H^1(X, \mathbb{R})) \otimes \det(\mathbb{H}^2_{+,g}(X)^\vee)$. Let $[\mathcal{W}^{\gamma}_{X,\beta}]_{\mathcal{O}} \in H_{w_c}(\mathcal{B}(c)^*, \mathbb{Z})$ be the fundamental class associated with the choice of $\mathcal{O}$.

Definition 6 *Let X be a closed connected oriented 4-manifold, $c \in H^2(X, \mathbb{Z})$ a characteristic element, (g, b) a c-good pair, $\mathfrak{c} \in \pi_0({\mathcal{C}}/_{\mathrm{Aut}(\hat{P})})$, and $\mathcal{O}$ an orientation of the line* $\det(H^1(X, \mathbb{R})) \otimes \det(\mathbb{H}^2_{+,g}(X)^\vee)$. *The corresponding Seiberg-Witten form is the element*

$$SW^{(g,b)}_{X,\mathcal{O}}(\mathfrak{c}) \in \Lambda^* H^1(X, \mathbb{Z})$$

defined by

$$SW^{(g,b)}_{X,\mathcal{O}}(\mathfrak{c})(l_1 \wedge \ldots \wedge l_r) = \left\langle \nu(l_1) \cup \ldots \cup \nu(l_r) \cup u^{\frac{w_c - r}{2}}, [\mathcal{W}^{\gamma}_{X,\beta}]_{\mathcal{O}} \right\rangle$$

for decomposable elements $l_1 \wedge \ldots \wedge l_r$ with $r \equiv w_c$ (mod 2) . Here γ lifts the pair $(g, \mathfrak{c})$ and $\beta \in b$ is a regular value of $p_{\gamma,b}$.

Remark: The form $SW^{(g,b)}_{X,\mathcal{O}}(\mathfrak{c})$ is well-defined, since the cohomology classes u, $\nu(l_i)$, as well as the trivialization of the orientation line bundle extend to the quotient ${}^{\mathcal{A}(\det\Sigma^+)\times(A^0(\Sigma^+)\setminus\{0\})\times b}/_{\mathcal{G}}$, and since, by Proposition 4 *iv)*, the homology class defined by $[\mathcal{W}^{\gamma}_{X,\beta}]_{\mathcal{O}}$ in this quotient depends only on $(g_\gamma,\mathfrak{c}_\gamma)$ and b.

Now there are two cases:

If $b_+>1$, then, by Proposition 4 *v), vi)*, the form $SW^{(g,b)}_{X,\mathcal{O}}(\mathfrak{c})$ is also independent of (g,b), since the cohomology classes and the trivialization of the orientation line bundle extend to $\mathrm{Aut}(\hat{P})$-invariant objects on the universal quotient ${}^{\mathcal{A}(\det\Sigma^+)\times(A^0(\Sigma^+)\setminus\{0\})\times Z^2_{\mathrm{DR}}(X)\times\mathcal{C}}/_{\mathcal{G}}$. Thus we may simply write $SW_{X,\mathcal{O}}(\mathfrak{c})\in\Lambda^*H^1(X,\mathbb{Z})$. If $b_1=0$, then we obtain numbers which we denote by $n^{\mathcal{O}}_{\mathfrak{c}}$; these numbers can be considered as refinements of the numbers $n^{\mathcal{O}}_c$ which were defined in [W]. Indeed, $n^{\mathcal{O}}_c=\sum_{\mathfrak{c}} n^{\mathcal{O}}_{\mathfrak{c}}$, the summation being over all $\mathfrak{c}\in\pi_0({}^{\mathcal{C}}/_{\mathrm{Aut}(\hat{P})})$.

Suppose now that $b_+=1$. There is a natural map $\mathcal{M}et_X\longrightarrow\mathbb{P}(H^2_{\mathrm{DR}}(X))$ which sends a metric g to the line $\mathbb{R}[\omega_+]\subset H^2_{\mathrm{DR}}(X)$, where ω_+ is any non-trivial g-selfdual harmonic form. Let $\mathbf{H}$ be the hyperbolic space

$$\mathbf{H}:=\{\omega\in H^2_{\mathrm{DR}}(X)|\ \omega^2=1\}\ .$$

$\mathbf{H}$ has two connected components, and the choice of one of them orients the lines $\mathbb{H}^2_{+,g}(X)$ for all metrics g. Furthermore, having fixed a component $\mathbf{H}_0$ of $\mathbf{H}$, every metric defines a unique g-self-dual form ω_g with $[\omega_g]\in\mathbf{H}_0$.

Definition 7 *Let X be a manifold with $b_+=1$, and let $c\in H^2(X,\mathbb{Z})$ be characteristic. The <u>wall</u> associated with c is the hypersurface $c^\perp:=\{(\omega,b)\in\mathbf{H}\times H^2_{\mathrm{DR}}(X)|\ (c-b)\cdot\omega=0\}$. The connected components of $\mathbf{H}\setminus c^\perp$ are called <u>chambers</u> of type c.*

Notice that the walls are non-linear! Every characteristic element c defines precisely four chambers of type c, namely

$$C_{\mathbf{H}_0,\pm}:=\{(\omega,b)\in\mathbf{H}_0\times H^2_{\mathrm{DR}}(X)|\ \pm(c-b)\cdot\omega<0\}\ ,$$

where $\mathbf{H}_0$ is one of the components of $\mathbf{H}$. Each of these four chambers contains pairs of the form $([\omega_g],b)$. Let $\mathcal{O}_1$ be an orientation of $H^1(X,\mathbb{R})$.

Definition 8 *The Seiberg-Witten invariant associated with the data* $(\mathcal{O}_1, \mathbf{H}_0, \mathfrak{c})$ *is the function*

$$SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c}) : \{\pm\} \longrightarrow \Lambda^* H^1(X,\mathbb{Z})$$

given by $SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})(\pm) := SW^{(g,b)}_{X,\mathcal{O}}(\mathfrak{c})$*, where* $\mathcal{O}$ *is the orientation defined by* $(\mathcal{O}_1, \mathbf{H}_0)$*, and* (g,b) *is a pair such that* $([\omega_g], b)$ *belongs to the chamber* $C_{\mathbf{H}_0,\pm}$*.*

Remark: The intersection $c^\perp \cap \mathbf{H} \times \{0\}$ defines a non-trivial wall in $\mathbf{H} \times \{0\}$ if and only if $c^2 < 0$. This means that invariants of index $w_c < \frac{b_2-10}{4} + b_1$ could also be defined for the chambers $C^{\pm}_{\mathbf{H}_0} := \{\omega \in \mathbf{H}_0 | \ \pm c \cdot \omega < 0\}$. However, when c is rationally non-zero and $c^2 \geq 0$, then $\mathbf{H}_0 \times \{0\}$ is entirely contained in one of the chambers $C_{\mathbf{H}_0,\pm}$.

Note that, changing the orientation $\mathcal{O}_1$ changes the invariant by a factor -1, and that $SW_{X,(\mathcal{O}_1,-\mathbf{H}_0)}(\mathfrak{c})(\pm) = -SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})(\mp)$.

Remark: A different approach - adapting ideas from intersection theory to construct "Seiberg-Witten multiplicities" - has been proposed by R. Brussee.

4 The wall crossing formula

In this section we prove a wall crossing formula for the complete Seiberg-Witten invariant $SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})$. This formula generalizes previous results of [W], [KM] and [LL]. Our proof is based on ideas similar to the ones in [LL], but our method - using the real bow up of the (singular) locus of reducible points as in [OT2] - has some advantages: It allows us to construct explicitely a smooth bordism to which the cohomology classes u, $\nu(l_i)$ extend in a natural way, and it enables us a to give a simple description of that part of its boundary which lies on the wall.

Our main result can be formulated by saying that the the difference $SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})(+) - SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})(-)$ is an abelian $Spin^c$-form. These abelian $Spin^c$-forms are topological invariants which will be introduced in the first subsection. Using a remark of [LL], we give an explicit formula for these invariants in the case of manifolds with $b_+ = 1$.

4.1 Abelian $Spin^c(4)$-forms

Let X be an oriented 4-manifold, $c \in H^2(X, \mathbb{Z})$ a characteristic element, and $\hat{P}$ a compatible $Spin^c(4)$-bundle. Let $\Sigma^{\pm}$ be the associated spinor bundles, define $L := \det \Sigma^{\pm}$, and put:

$$\mathcal{B}(L) := {}^{\mathcal{A}(L)}\!/_{\mathcal{G}_0}\ ,\quad \mathcal{B}'(L) := {}^{\mathcal{A}(L)}\!/_{\mathcal{G}_0^2}\ .$$

There is an obvious covering projection

$$s : \mathcal{B}'(L) \longrightarrow \mathcal{B}(L)$$

with fiber ${}^{\mathcal{G}_0}\!/_{\mathcal{G}_0^2} = {}^{H^1(X,\mathbb{Z})}\!/_{2H^1(X,\mathbb{Z})}$.

The homotopy equivalence $\mathcal{G}_0 \xrightarrow{\simeq} H^1(X, \mathbb{Z})$ induces canonical isomorphisms

$$\mu : \Lambda^* \left({}^{H_1(X,\mathbb{Z})}\!/_{\mathrm{Tors}} \right) \xrightarrow{\simeq} H^*(\mathcal{B}(L), \mathbb{Z})\ ,$$

$$\nu' : \Lambda^* \left({}^{H_1(X,\mathbb{Z})}\!/_{\mathrm{Tors}} \right) \xrightarrow{\simeq} H^*(\mathcal{B}'(L), \mathbb{Z})\ ,$$

such that $s^* \circ \mu = 2\nu'$.

Now fix a metric g and let h_c be the harmonic representant of the de Rham class $c_{\mathrm{DR}} \in H^2_{\mathrm{DR}}(X)$. The spaces $\mathcal{B}(L)$, $\mathcal{B}'(L)$ are trivial fibre bundles over the affine subspace $h_c^+ + d^+(A^1) = h_c^+ + (\mathbb{H}^2_{g,+}(X))^{\perp} \subset A^2_+$ via the maps induced by $A \longmapsto \frac{i}{2\pi} F_A^+$.

For a given 2-form $\beta \in h_c^+ + d^+(A^1)$ let

$$\mathcal{T}_\beta(L) \subset \mathcal{B}(L)\ ,\quad \mathcal{T}'_\beta(L) \subset \mathcal{B}'(L)$$

be the fibers of these maps over β. These fibers are tori, consisting of equivalence classes of solutions of of the equation

$$F_A^+ + 2\pi i \beta = 0$$

modulo $\mathcal{G}_0$ respectively $\mathcal{G}_0^2$. Indeed, the choice of a solution $A_0 \in \mathcal{A}(L)$ yields identifications

$$\mathcal{T}_\beta(L) = {}^{H^1(X,\mathbb{R})}\!/_{H^1(X,\mathbb{Z})}\ ,\quad \mathcal{T}'_\beta(L) = {}^{H^1(X,\mathbb{R})}\!/_{2H^1(X,\mathbb{Z})}\ .$$

Now fix a Clifford map $\gamma \in \mathcal{C}$ of type $\hat{P}$ with $g_\gamma = g$, and a 2-form $\beta \in h_c^+ + d^+(A^1)$. Let $S(A^0(\Sigma^+))$ be the unit sphere in $A^0(\Sigma^+)$ with respect to the L^2-norm.

Definition 9 *The β-twisted $Spin^c(4)$-equations for a pair $(A, \Phi) \in \mathcal{A}(L) \times S(A^0(\Sigma^+))$ are the equations*

$$\left\{ \begin{array}{lcl} \not{D}_A \Phi & = & 0 \\ F_A^+ + 2\pi i \beta & = & 0 \ . \end{array} \right. \qquad (S_\beta^\gamma)$$

The gauge group $\mathcal{G}$ acts on $\mathcal{A}(L) \times S(A^0(\Sigma^+))$ by $(A, \Phi) \cdot f = (A^{f^2}, f^{-1}\Phi)$, letting invariant the space of solutions of (S_β^γ). We denote by $\hat{\mathcal{T}}_\beta'^\gamma(L)$ the moduli space of solutions; it is a projective fiber space over the "Brill-Noether locus" in $\mathcal{T}_\beta'(L)$. Let

$$q : {\mathcal{A}(L) \times S(A^0(\Sigma^+))}/_{\mathcal{G}} \longrightarrow \mathcal{B}'(L)$$

be the natural projection.

Lemma 10 *There is a natural isomorphism*

$$\mathbb{Z}[u] \otimes \Lambda^* \left({H_1(X, \mathbb{Z})}/_{\mathrm{Tors}} \right) \longrightarrow H^* \left({\mathcal{A}(L) \times S(A^0(\Sigma^+))}/_{\mathcal{G}}, \mathbb{Z} \right)$$

whose restriction to $\Lambda^* \left({H_1(X, \mathbb{Z})}/_{\mathrm{Tors}} \right)$ factors as $q^* \circ \nu'$. The class u restricts to the positive generator of the second cohomology group $\mathbb{P}(A^0(\Sigma^+))$ of the fibers of q.

This lemma, as well as the following proposition can be proved using the same methods as in the proofs of Lamma 5 and Proposition 4.

Set $\delta_c := \frac{1}{8}(c^2 - \sigma(X))$.

Proposition 11 *For generic elements $\beta \in h_c^+ + d^+(A^1)$, the moduli space $\hat{\mathcal{T}}_\beta'^\gamma(L)$ is a closed smooth manifold of dimension $b_1 + 2\delta_c - 2$. It can be oriented by choosing an orientation $\mathcal{O}_1$ of $H^1(X, \mathbb{R})$. The fundamental class*

$$[\hat{\mathcal{T}}_\beta'^\gamma(L)]_{\mathcal{O}_1} \in H_{b_1 + 2\delta_c - 2} \left({\mathcal{A}(L) \times S(A^0(\Sigma^+))}/_{\mathcal{G}}, \mathbb{Z} \right)$$

depends only on the component $\mathfrak{c}_\gamma \in \pi_0 \left({\mathcal{C}}/_{\mathrm{Aut}(\hat{P})} \right)$.

We can now define the abelian $Spin^c(4)$-forms.

Definition 12 *Let X be a closed connected oriented 4-manifold, and $\mathcal{O}_1$ an orientation of $H^1(X,\mathbb{R})$. Let $c \in H^2(X,\mathbb{Z})$ be a characteristic element and $\mathfrak{c} \in \pi_0\left(\mathcal{C}/_{\mathrm{Aut}(\hat{P})}\right)$. The corresponding $Spin^c(4)$-form is the element $\sigma_{X,\mathcal{O}_1}(\mathfrak{c}) \in \Lambda^*(H^1(X,\mathbb{Z}))$ defined by the formula*

$$\sigma_{X,\mathcal{O}_1}(\mathfrak{c})(l_1 \wedge \ldots \wedge l_r) := \langle \nu'(l_1) \cup \ldots \cup \nu'(l_r) \cup u^{\frac{b_1+2\delta_c-2-r}{2}}, [\hat{\mathcal{T}}_\beta'^{\gamma}(L)]_{\mathcal{O}_1}\rangle$$

for decomposable elements $l_1 \wedge \ldots \wedge l_r$ with $r \equiv b_1$ (mod 2). Here γ induces $\mathfrak{c} = \mathfrak{c}_\gamma$ and β is generic.

Note that the expected dimension $b_1 + 2\delta_c - 2$ of the moduli space $\hat{\mathcal{T}}_\beta'^{\gamma}(L)$ coincides with w_c iff $b_+ = 1$. The $Spin^c$-forms $\sigma_{X,\mathcal{O}_1}(\mathfrak{c})$ are topological invariants, which can be explicitely computed. This is our next aim.

Let $pr_2 : \mathcal{A}(L) \times X \longrightarrow X$ be the projection onto the second factor, and put

$$\mathbb{P} := {pr_2^*(\hat{P})}/_{\mathcal{G}_0} ,$$

where $\mathcal{G}_0$ acts on $pr_2^*(\hat{P}) = \mathcal{A}(L) \times \hat{P}$ by $(A, \hat{p}) \cdot f = (A^{f^2}, f(\hat{p}))$. The universal $Spin^c(4)$-bundle $\mathbb{P}$ over $\mathcal{B}'(L) \times X$ comes with a tautological connection $\mathbb{A}$ in the X-direction. Let $\mathbb{L} := {pr_2^*(L)}/_{\mathcal{G}_0}$ be the universal line bundle over $\mathcal{B}(L) \times X$; its pull back $(s \times \mathrm{id})^*(\mathbb{L})$ to $\mathcal{B}'(L) \times X$ is the universal determinant bundle $\mathbb{P} \times_{\det} \mathbb{C}$. The Chern class $c_1(\mathbb{L})$ has a Künneth decomposition $c_1(\mathbb{L}) = 1 \otimes c + c_1(\mathbb{L})^{1,1}$ whose $(1,1)$-component $c_1(\mathbb{L})^{1,1} \in H^1(\mathcal{B}(L),\mathbb{Z}) \otimes H^1(X,\mathbb{Z})$, considered as homomorphism $\mu_1 : {H^1(X,\mathbb{Z})}/_{\mathrm{Tors}} \longrightarrow H^1(\mathcal{B}(L),\mathbb{Z})$, is given by the restriction of the isomorphism μ.

We fix a basis $(l_i)_{1 \le i \le b_1}$ of ${H_1(X,\mathbb{Z})}/_{\mathrm{Tors}}$ and let l^i be the elements of the dual basis. Then

$$c_1(\mathbb{L}) = 1 \otimes c + \sum_{i=1}^{b_1} \mu(l_i) \otimes l^i ,$$

$$c_1(\mathbb{P} \times_{\det} \mathbb{C}) = 1 \otimes c + 2\sum_{i=1}^{b_1} \nu'(l_i) \otimes l^i .$$

Let $\gamma : \Lambda^1 \longrightarrow \hat{P} \times_\pi \mathbb{R}^4$ be a Clifford map and $\boldsymbol{\gamma} : pr_2^*(\Lambda^1) \longrightarrow \mathbb{P} \times_\pi \mathbb{R}^4$ the induced isomorphism. We restrict the universal objects $(\mathbb{P}, \boldsymbol{\gamma}, \mathbb{A})$ to the subspace $\mathcal{T}'_\beta(L)\times X$ and denote the restrictions by the same symbols. The Chern character of the virtual bundle $index(\not{D}_{\mathbb{A}})$ over the torus $\mathcal{T}'_\beta(L)$ is

$$ch(index(\not{D}_{\mathbb{A}}))=\left[\left(e^{\frac{c}{2}+\sum\nu'(l_i)\otimes l^i}\right)\cup\left(1-\frac{1}{24}p_1(X)\right)\right]/[X]=$$

$$=\left(e^{\frac{c}{2}+\sum\nu'(l_i)\otimes l^i}\right)/[X]-\frac{1}{8}\sigma(X)=$$

$$=-\frac{1}{8}\sigma(X)+\frac{1}{8}c^2+\frac{1}{3!}(\frac{c}{2}+\sum\nu'(l_i)\otimes l^i)^3/[X]+\frac{1}{4!}(\frac{c}{2}+\sum\nu'(l_i)\otimes l^i)^4/[X]\ .$$

To simplify this expression, put $c_{ij}:=\langle c\cup l^i\cup l^j,[X]\rangle$, $l_{hijk}:=\langle l^h\cup l^i\cup l^j\cup l^k,[X]\rangle$. The numbers c_{ij} are even since c is characteristic and $(l^i\cup l^j)^2=0$. Substituting into the formula above we find:

$$\begin{aligned}ch(index(\not{D}_{\mathbb{A}}))=\ &-\tfrac{1}{8}\sigma(X)+\tfrac{1}{8}c^2+\tfrac{1}{2}\sum_{i<j}c_{ij}\ [\nu'(l_i)\cup\nu'(l_j)]+\\ &+\sum_{h<i<j<k}l_{hijk}[\nu'(l_h)\cup\nu'(l_i)\cup\nu'(l_j)\cup\nu'(l_k)]\ .\end{aligned}$$

The cohomology class

$$u_c:=\frac{1}{2}\sum_{i<j}c_{ij}\ [\nu'(l_i)\cup\nu'(l_j)]\in H^2(\mathcal{T}'_\beta(L),\mathbb{Z})$$

has the following invariant description: The assignment

$$(a,b)\longmapsto\frac{1}{2}\langle c\cup a\cup b,[X]\rangle$$

defines a $\mathbb{Z}$-valued skew-symmetric bilinear form on $H^1(X,\mathbb{Z})$; using the isomorphism $H^1(X,\mathbb{Z})\simeq 2H^1(X,\mathbb{Z})$, we get a cohomology class in $H^2(\mathcal{T}'_\beta(L),\mathbb{Z})$ $=\Lambda^2(2H^1(X,\mathbb{Z})^\vee)$, and this coincides with u_c. Clearly $u_c=c_1(index(\not{D}_{\mathbb{A}}))$.

Lemma 13 *[LL] Let X be a 4-manifold with with $b_+=1$. Then*

$$c_k(index(\not{D}_{\mathbb{A}}))=\frac{1}{k!}u_c^k\ .$$

Proof: In the case $b_+=1$, all coefficients l_{hijk} vanish, so $ch_k(index(\not{D}_{\mathbb{A}}))=0$ for $k\geq 2$. ■

Regard now $u_c\in H^2(\mathcal{T}'_\beta(L),\mathbb{Z})$ as an element of $\Lambda^2\left(H_1(X,\mathbb{Z})/_{\mathrm{Tors}}\right)$.

Proposition 14 *Let X be a closed connected oriented 4-manifold with $b_+ = 1$, and $\mathcal{O}_1$ an orientation of $H^1(X,\mathbb{R})$. Let $c \in H^2(X,\mathbb{Z})$ be a characteristic element, $\hat{P}$ a compatible $Spin^c(4)$-bundle and $\mathfrak{c} \in \pi_0\left({\mathcal{C}}/{\mathrm{Aut}(\hat{P})}\right)$. Choose the generator $l_{\mathcal{O}_1} \in \Lambda^{b_1}(H^1(X,\mathbb{Z}))$ which defines the orientation $\mathcal{O}_1$. For every $\lambda \in \Lambda^r\left({H_1(X,\mathbb{Z})}/{\mathrm{Tors}}\right)$ with $r \equiv b_1 \pmod 2$ and $0 \le r \le \min(b_1, w_c)$, we have:*

$$\sigma_{X,\mathcal{O}_1}(\mathfrak{c})(\lambda) = \langle \lambda \wedge \exp(-u_c), l_{\mathcal{O}_1}\rangle \ .$$

In all other cases $\sigma_{X,\mathcal{O}_1}(\mathfrak{c})(\lambda) = 0$.

The proof follows from a more general result which we will now explain.

Let T be a closed connected oriented manifold. Consider Hilbert vector bundles E and F over T, and a smooth family of Fredholm operators $q_t : E_t \longrightarrow F_t$ of constant index δ. Suppose the map $\tilde{q} : E \setminus \{0\} \longrightarrow F$ is a submersion, so that its zero locus $\tilde{T} := Z(\tilde{q})$ is a (finite dimensional) manifold which fibers over the possibly singular "Brill-Noether locus" $BN_q := \{t \in T | \ \ker q_t \neq \{0\}\}$ of the family q. Put $\hat{T} := {\tilde{T}}/{\mathbb{C}^*}$. The projection $\hat{p} : \hat{T} \longrightarrow T$ induces a projective fibration over BN_q. Note that $\hat{T}$ comes with a canonical cohomology class $u \in H^2(\hat{T},\mathbb{Z})$ induced by the dual of the $\mathbb{C}^*$-bundle $\tilde{T} \longrightarrow \hat{T}$; the restriction of u to any fiber $\hat{p}^{-1}(t) = \mathbb{P}(\ker q_t)$ is the positive generator of the second cohomology group of this projective space. We wish to calculate the direct images $\hat{p}_*(u^k)$ for all $k \in \mathbb{N}$, $k \ge \delta$ in terms of the Chern classes of the (virtual) index bundle *index*(q) of the family q. In the particular case of Dirac operators on a 4-manifold with $b_+ = 1$, similar computations have been carried out in [LL] .

Proposition 15 *Let $c_i = c_i(index(q))$ be the Chern classes of $index(q)$, and define polynomials $(p_k)_{k\ge\delta-1}$ by the recursive relations:*

$$p_{\delta-1} = 1,\ p_k = -\sum_{i=1}^{k-\delta+1} c_i p_{k-i} \ .$$

For every non-negative integer $k \ge \delta$ we have

$$\hat{p}_*(u^k) = p_k(c_1, c_2, \ldots) \ ,$$

hence $\hat{p}_*(u^{\delta-1}) = 1 \in H^0(T,\mathbb{Z})$ when $\delta > 0$.

Proof: One can find a smooth family of Fredholm operators $(Q_t)_{t\in T}$, $Q_t : E_t \oplus \mathbb{C}^n \longrightarrow F_t$ with $Q_t|_{E_t\times\{0\}} = q_t$, such that Q_t is surjective of positive index $n+\delta$ for every $t \in T$. The associated map $\tilde{Q} : (E \oplus \mathbb{C}^n) \setminus \{0\} \longrightarrow F$ is a submersion and $Z(\tilde{Q})$ is a locally trivial fiber bundle over T with standard fiber $\mathbb{C}^{n+\delta} \setminus \{0\}$. Indeed, $Z(\tilde{Q})$ is the complement of the zero section of the vector bundle

$$V := \bigcup_{t\in T} \ker Q_t \ .$$

The space $\tilde{T}$ can be identified with the zero locus $Z(\zeta)$, of the map

$$\zeta : Z(\tilde{Q}) \longrightarrow \mathbb{C}^n$$

given by $\zeta(e, z) = z$. The map ζ is a submersion in all points of $\tilde{T}$, since $\tilde{q}$ was such.

Let $p_V : \mathbb{P}(V) \longrightarrow T$ be the obvious projection, and denote by $U \in H^2(\mathbb{P}(V), \mathbb{Z})$ the Chern class of the dual of the tautological bundle. The map $\hat{p} : \hat{T} \longrightarrow T$ factors through the inclusion $j : \hat{T} \hookrightarrow \mathbb{P}(V)$, and the fundamental class $j_*[\hat{T}]$ is Poincaré dual to U^n. Therefore we have

$$\hat{p}_*(u^k) = [PD_T^{-1} \circ p_{V*} \circ j_* \circ PD_{\hat{T}}](u^k) = [PD_T^{-1} \circ p_{V*} \circ PD_{\mathbb{P}(V)}](U^{k+n}) = p_{V*}(U^{k+n}).$$

Since $index(q) = [V] - [\mathbb{C}^n] \in K(T)$, we have $c_i(V) = c_i(index(q)) = c_i$, and therefore

$$U^{\delta+n} = -\sum_{i=1}^{\delta+n} p_V^*(c_i) U^{\delta+n-i} \ .$$

Multiplying with $U^{k-\delta}$ and using $p_{V*}(U^{\delta+n-1}) = 1$, we get the recursion relations

$$p_{V*}(U^{k+n}) = -\sum_{i=1}^{\delta+n} p_{V*}(U^{k+n-i}) c_i \ ,$$

hence $\hat{p}_{V*}(U^{k+n}) = p_k$ for $k \geq \delta - 1$ by induction. ■

Now we can prove Proposition 14 by applying the result above to the map $\hat{p}\hat{\mathcal{T}}_\beta'^{\gamma}(L) \longrightarrow \mathcal{T}_\beta'(L)$ and the family $\not{D}_{\mathbb{A}}$ of Dirac operators over $\mathcal{T}_\beta'(L)$.

Since $c_k(index(\not{D}_{\mathbb{A}})) = \frac{1}{k!} u_c^k$, we get $p_*(u^{\delta-1+k}) = p_{\delta-1+k} = \frac{(-1)^k}{k!} u_c^k$, hence

$$\hat{p}_*(u^{\frac{w_c - r}{2}}) = \frac{(-1)^{\left[\frac{b_1 - r}{2}\right]}}{\left[\frac{b_1 - r}{2}\right]!} \ u_c{}^{\left[\frac{b_1 - r}{2}\right]}$$

for any non-negative integer r with $r \leq \min(b_1, w_c)$ and $r \equiv b_1 \pmod 2$.

Therefore, for every $\lambda \in \Lambda^r \left({H_1(X,\mathbb{Z})}/_{\mathrm{Tors}} \right)$, we find

$$\begin{aligned}
\sigma_{X,\mathcal{O}_1}(\mathfrak{c})(\lambda) &= \langle \hat{p}^* \nu'(\lambda) \cup u^{\frac{w_c - r}{2}}, [\hat{\mathcal{T}}_\beta'^\gamma(L)]_{\mathcal{O}_1} \rangle = \langle \nu'(\lambda) \cup \hat{p}_*(u^{\frac{w_c-r}{2}}), [\mathcal{T}'_\beta(L)]_{\mathcal{O}_1} \rangle \\
&= \frac{(-1)^{\left[\frac{b_1 - r}{2}\right]}}{\left[\frac{b_1-r}{2}\right]!} \left\langle \lambda \wedge u_c{}^{\left[\frac{b_1-r}{2}\right]}, l_{\mathcal{O}_1} \right\rangle \ ,
\end{aligned}$$

which proves the proposition. ∎

4.2 Wall crossing

The following theorem generalizes results of [W], [KM] and [LL].

Theorem 16 *Let X be a closed connected oriented 4-manifold with $b_+ = 1$, and $\mathcal{O}_1$ an orientation of $H^1(X,\mathbb{R})$. For every class $\mathfrak{c}$ of $Spin^c(4)$-structures of Chern class c and every component $\mathbf{H}_0$ of $\mathbf{H}$, the following holds:*

$$SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})(+) - SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})(-) = \sigma_{X,\mathcal{O}_1}(\mathfrak{c}) \ .$$

We need some preparations before we can prove the theorem.

Fix a compatible $Spin^c(4)$-bundle $\hat{P}$ with spinor bundles $\Sigma^\pm$ and determinant $L := \det \Sigma^\pm$. Choose a Clifford map γ with $\mathfrak{c}_\gamma = \mathfrak{c}$, set $g = g_\gamma$, and let ω_g be the generator of $\mathbb{H}^2_{+,g}(X)$ whose class belongs to $\mathbf{H}_0$. Put $s_c := c \cdot [\omega_g]$, so that the g-harmonic representant of $c - s_c[\omega_g]$ is g-anti-selfdual.

Consider first a cohomology class $b_0 \in H^2_{\mathrm{DR}}(X)$ with $(c - b_0)\cdot[\omega_g] = 0$, and let $\beta_0 \in b_0$. The moduli space $\mathcal{W}_{\beta_0} := \mathcal{W}^\gamma_{X,\beta_0}$ contains the closed subset of reducible solutions of the form $(A, 0)$, where A solves the equation

$$F_A^+ + 2\pi i \beta_0^+ = 0 \ .$$

This closed subspace can be identified with $\mathcal{T}'_{\beta_0^+}(L)$.

Consider the equations

$$\left\{ \begin{array}{lcl} \not{D}_A \Psi &=& 0 \\ \Gamma(F_A^+ + 2\pi i \beta^+) &=& 2(\Psi\bar{\Psi})_0 \end{array} \right. \qquad (SW^\gamma)$$

for a triple $(A, \Psi, \beta) \in \mathcal{A}(L) \times A^0(\Sigma^+) \times Z^2_{\mathrm{DR}}(X)$, denote by $\mathcal{W}$ the corresponding moduli space of solutions, and by $p : \mathcal{W} \longrightarrow Z^2_{\mathrm{DR}}(X)$ the natural projection. $\mathcal{W}$ is singular in the points of the form $[A, 0, \beta]$. If such a triple is a solution, then $(c - [\beta]) \cdot [\omega_g] = 0$, and the singular part of $\mathcal{W}$ is

$$\mathcal{S} = \bigcup_{(c-[\beta])\cdot[\omega_g]} \mathcal{T}'_{\beta^+}(L) \ .$$

Now perform a "real blow up" of the singular locus $\mathcal{S} \subset \mathcal{W}$ in the Ψ-direction. This means, consider the equations

$$\left\{ \begin{array}{lcl} \not{D}_A \Phi & = & 0 \\ \Gamma(F_A^+ + 2\pi i \beta^+) & = & 2t(\Phi\bar{\Phi})_0 \end{array} \right. \qquad (S\hat{W}^\gamma)$$

for a tuple $(A, \Phi, t, \beta) \in \mathcal{A}(L) \times S(A^0(\Sigma^+)) \times \mathbb{R} \times Z^2_{\mathrm{DR}}(X)$, where $S(A^0(\Sigma^+))$ is the unit sphere in $A^0(\Sigma^+)$ with respect to the L^2-norm. Denote by $\hat{\mathcal{W}}$ the moduli space of solutions of $(S\hat{W}^\gamma)$ and by $q_{\mathbb{R}}$, q the natural projections on $\mathbb{R}$ and $Z^2_{\mathrm{DR}}(X)$ respectively. Let $\hat{\mathcal{W}}^{\geq 0} := q_{\mathbb{R}}^{-1}(\mathbb{R}_{\geq 0})$ be the closed subset defined by the inequality $t \geq 0$, and, for a form $\beta \in Z^2_{\mathrm{DR}}(X)$, put $\hat{\mathcal{W}}_\beta := q^{-1}(\beta)$ and $\hat{\mathcal{W}}_\beta^{\geq 0} = \hat{\mathcal{W}}_\beta \cap \hat{\mathcal{W}}^{\geq 0}$.

There is a natural map $\hat{\mathcal{W}}^{\geq 0} \xrightarrow{\rho} \mathcal{W}$, given by $(A, \Phi, t, \beta) \longmapsto (A, t^{\frac{1}{2}}\Phi, \beta)$, which contracts the locus $\hat{\mathcal{S}} := \{t = 0\}$ to $\mathcal{S}$ and defines a real analytic isomorphism $\hat{\mathcal{W}}^{\geq 0} \setminus \hat{\mathcal{S}} \longrightarrow \mathcal{W} \setminus \mathcal{S}$. Note that

$$\hat{\mathcal{S}} = \bigcup_{(c-[\beta])\cdot[\omega_g]} \hat{\mathcal{T}}'_{\beta^+}(L) \ ,$$

where $\hat{\mathcal{T}}'_{\beta^+}(L)$ is a projective fibration over the Brill-Noether locus in $\mathcal{T}'_{\beta^+}(L)$.

Lemma 17 *If $(g, [\beta])$ is c-good, then $q_{\mathbb{R}}|_{\hat{\mathcal{W}}_\beta^{\geq 0}}$ is bounded below by a positive number. The space $\hat{\mathcal{W}}_\beta^{\geq 0}$ is open and closed in $\hat{\mathcal{W}}_\beta$, and it is isomorphic to $\mathcal{W}_\beta$ via the map ρ.*

Proof: If $\hat{\mathcal{W}}_\beta$ would contain a sequence $[(A_n, \Phi_n, t_n, \beta)]_{n\in\mathbb{N}}$ with $t_n \searrow 0$, then, by Proposition 1 *i*), we could find a subsequence $(m_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ such that $(A_n, t_n^{\frac{1}{2}}\Phi_n)_{n\in\mathbb{N}}$ converges to a point in $\mathcal{W}_\beta$. This point must be reducible. ■

Lemma 18 *$\hat{\mathcal{W}}$ is a smooth manifold .*

Proof: Let (Ψ^-, S) be orthogonal to $\mathrm{im} d_\tau(\hat{F})$, where $\hat{F} = (\hat{F}_1, \hat{F}_2)$ is the map given by the left-hand side of $(\hat{SW^\gamma})$, and $\hat{\tau} = (A, \Phi, t, \beta)$ is a point in $\mathcal{A}(L) \times S(A^0(\Sigma^+)) \times \mathbb{R} \times Z^2_{\mathrm{DR}}(X)$. Using variations of β, we get $i\Gamma^{-1}(S) \in d^*(A^3) \cap A^2_+$, hence $S = 0$. Using now variations of A and the non-triviality of Φ, we get $\Psi^- = 0$. ■

Lemma 19 *The linear map $h : A^1 \times \mathbb{R} \longrightarrow Z^2_{\mathrm{DR}}(X)$, given by $h(\alpha, s) = s\omega_g + d\alpha$, is transverse to q.*

Proof: The image of dh is the subspace $d(A^1) \oplus \mathbb{H}^2_{+,g}(X)$ of $Z^2_{\mathrm{DR}}(X)$ which coincides with the orthogonal complement of $\mathbb{H}^2_{-,g}(X)$ in $Z^2_{\mathrm{DR}}(X)$. It suffices to show that $\mathbb{H}^2_{-,g}(X)$ is contained in the image of $d_{\hat{\tau}}q$, for every $\hat{\tau} \in \hat{\mathcal{W}}$. But if $\hat{\tau} = (A, \Phi, t, \beta)$ solves the equations $(S\hat{W}^\gamma)$, then also $\hat{\tau}_t := (A, \Phi, t, \beta + r\omega_-)$ is a solution for every $\omega_- \in \mathbb{H}_{-,g}(X)$ and every $r \in \mathbb{R}$. Therefore $\mathbb{H}_{-,g}(X)$ is contained in the image of $d_{\hat{\tau}}q$. ■

Since h is transverse to q, the fibre product $\mathcal{V} := (A^1 \times \mathbb{R}) \times_{(h,q)} \hat{\mathcal{W}}$ is a smooth manifold, and, putting $h_\alpha := h(\alpha, \cdot)$, we see that for any α in a second category subset of A^1, the fibre product $\mathcal{V}_\alpha := \mathbb{R} \times_{(h_\alpha, p)} \hat{\mathcal{W}}$ is a smooth submanifold of $\mathcal{V}$.

Lemma 20 *The map $\theta : \mathcal{V} \longrightarrow \mathbb{R}$, projecting $(\alpha, s, A, \Phi, t, s\omega_g + d\alpha)$ to t, is a submersion. For any α in a second category subset of A^1, the restricted map $\theta|_{\mathcal{V}_\alpha}$ is a submersion in all points of $Z(\theta) \cap \mathcal{V}_\alpha$.*

Proof: $\mathcal{V}$ can be identified with the moduli space of tuples $(\alpha, s, A, \Phi, t) \in A^1 \times \mathbb{R} \times \mathcal{A}(L) \times S(A^0(\Sigma^+)) \times \mathbb{R}$ satisfying the equations

$$\left\{ \begin{array}{ccc} \not{D}_A\Phi & = & 0 \\ \Gamma(F_A^+ + 2\pi i(s\omega_g + d^+\alpha)) - 2t(\Phi\bar{\Phi})_0 & = & 0 \ . \end{array} \right.$$

Since the map defined by the left hand side of this system is a submersion in every tuple solving the equations, it suffices to show that the map

$$T : A^1 \times \mathbb{R} \times \mathcal{A}(L) \times S(A^0(\Sigma^+)) \times \mathbb{R} \longrightarrow A^0(\Sigma^-) \times A^0(su(\Sigma^+)) \times \mathbb{R} \ ,$$

defined by

$$T(\alpha, s, A, \Phi, t) = \left(\begin{array}{c} \not{D}_A\Phi \\ \Gamma(F_A^+ + 2\pi i(s\omega_g + d^+\alpha)) - 2t(\Phi\bar{\Phi})_0 \\ t \end{array} \right) \ ,$$

is a submersion in every point $v = (\alpha, s, A, \Phi, t)$ with $[(\alpha, s, A, \Phi, t)] \in \mathcal{V}$. If (Ψ^-, S, r) is orthogonal to $\mathrm{im}(d_v T)$, use first variations of s and α to get $S = 0$, then variations of A to get $\Psi^- = 0$, and then variations of t to get $r = 0$.

The second assertion follows by applying Sard's Theorem to the projection $Z(\theta) \longrightarrow A^1$ onto the first factor. ■

Lemma 21 *For any α in a second category subset of A^1, the moduli space $\hat{\mathcal{T}}'_{(s_c\omega + d^+\alpha)}(L)$ is a smooth manifold.*

Proof: If $(\alpha, s, A, \Phi, 0) \in Z(\theta)$, then the pair $(g, [s\omega_g + d\alpha])$ cannot be c-good, hence the s-component of every point in $Z(\theta)$ must be s_c. Thus there is a natural identification $\hat{\mathcal{T}}'_{(s_c\omega + d^+\alpha)}(L) = Z(\theta) \cap \mathcal{V}_\alpha$. ■

Since a pair $([\omega_g], [h(\alpha, s)])$ belongs to the wall $c^\perp$ iff $s = s_c$, every point $(\alpha, s, A, \Phi, t, s\omega_g + d\alpha) \in \mathcal{V}$ with $s \neq s_c$ must have a non-vanishing t-component.

Lemma 22 *The map $\chi : \mathcal{V} \longrightarrow \mathbb{R}$, projecting $(\alpha, s, A, \Phi, t, s\omega_g + d\alpha)$ to s, is a submersion in every point of $\mathcal{V} \setminus Z(\theta)$, in particular in all points $(\alpha, s, A, \Phi, t, s\omega_g + d\alpha)$ with $s \neq s_c$.*

Proof: It suffices to show that the map

$$U : A^1 \times \mathbb{R} \times \mathcal{A}(L) \times S(A^0(\Sigma^+)) \times \mathbb{R} \longrightarrow A^0(\Sigma^-) \times A^0(su(\Sigma^+)) \times \mathbb{R} \ ,$$

defined by

$$U(\alpha, s, A, \Phi, t) = \begin{pmatrix} \not{D}_A \Phi \\ \Gamma(F_A^+ + 2\pi i(s\omega_g + d^+\alpha)) - 2t(\Phi\bar{\Phi})_0 \\ s \end{pmatrix} \ ,$$

is a submersion in every point $v = (\alpha, s, A, \Phi, t)$ with $t \neq 0$. If (Ψ^-, S, r) is orthogonal to $\mathrm{im}(d_v U)$, we first use first variations of α to see that S is orthogonal to $d^+(A^1)$, then variations of A to get $\Psi^- = 0$, and then variations of Φ and t ($t \neq 0$!) to get $S = 0$. Finally, using variations of s one finds $r = 0$. ■

Applying Sard's theorem again, we have

Lemma 23 *Let S_0 be a countable subset of $\mathbb{R} \setminus \{s_c\}$. Then, for every α in a second category subset of A^1, the restricted map $\chi|_{\mathcal{V}_\alpha}$ is a submersion in every point of $\bigcup_{s \in S_0} Z(\chi|_{\mathcal{V}_\alpha} - s)$*

Now we can prove the theorem.

Proof: Fix $\alpha \in A^1$ such that $\theta|_{\mathcal{V}_\alpha}$ is a submersion in every point of $Z(\theta|_{\mathcal{V}_\alpha})$ and $\chi|_{\mathcal{V}_\alpha}$ is a submersion in every point of $Z(\chi|_{\mathcal{V}_\alpha} - (s_c \pm 1))$. Set $\beta_0 = h_\alpha(s_c) = s_c\omega_g + d\alpha$, $b_0 := [\beta_0] = [s_c\omega_g]$, and $\beta_\pm := h(\alpha, s_c \pm 1)$, $b_\pm := [\beta_\pm]$. Then $([\omega_g], b_0)$ belongs to the wall $c^\perp$, and the intersections $\hat{\mathcal{W}}_{\beta_\pm} := \hat{\mathcal{W}} \cap \{\beta = \beta_\pm\}$ are smooth. Note that $([\omega_g], b_\pm) \in C_{\mathbf{H}_0,\pm}$.

The space

$$\bar{\mathcal{V}}^{\geq 0} := \mathcal{V}_\alpha \cap \{s_c - 1 \leq s \leq s_c + 1\} \cap \{t \geq 0\} = [s_c - 1, s_c + 1] \times_{(h_\alpha, p)} \hat{\mathcal{W}}^{\geq 0}$$

is a smooth manifold with boundary $\hat{\mathcal{W}}^{\geq 0}_{\beta_+} \bigcup \hat{\mathcal{W}}^{\geq 0}_{\beta_-} \bigcup \hat{\mathcal{T}}'_{\beta_0^+}(L)$ which is isomorphic to $\mathcal{W}_{\beta_+} \bigcup \mathcal{W}_{\beta_-} \bigcup \hat{\mathcal{T}}'_{\beta_0^+}(L)$ according to Lemma 17.

Indeed, by the choice of α, $\mathcal{V}_\alpha$ is smooth, the projection on the t component is a submersion in all points of $\hat{\mathcal{T}}'_{\beta_0^+}(L)$ (Lemma 20), and the projection on the s-component is a submersion in the points of $\hat{\mathcal{W}}^{\geq 0}_{\beta_\pm}$ (Lemma 22) .

Now use the orientation $\mathcal{O}_1$ of $H^1(X,\mathbb{R})$ and the component $\mathbf{H}_0$ of $\mathbf{H}$ to endow the smooth moduli spaces $\hat{\mathcal{W}}^{\geq 0}_{\beta_\pm} = \mathcal{W}_{\beta_\pm}$ with the corresponding orientations. The manifold with boundary $\bar{\mathcal{V}}^{>0}$ can be oriented by $\mathcal{O}_1$, $\mathbf{H}_0$, and by choosing the natural orientation of the s-coordinate. Then the oriented boundary of $\bar{\mathcal{V}}^{>0}$ is

$$\partial\bar{\mathcal{V}}^{>0} = \hat{\mathcal{W}}^{\geq 0}_{\beta_+} \bigcup \left(-\hat{\mathcal{W}}^{\geq 0}_{\beta_-}\right) \ .$$

Recall that the moduli space $\hat{\mathcal{T}}'_{\beta_0^+}(L)$ could be oriented using <u>only</u> the orientation $\mathcal{O}_1$ of $H^1(X,\mathbb{R})$.

To determine the sign of the part $\hat{\mathcal{T}}'_{\beta_0^+}(L)$ of the oriented boundary $\partial\bar{\mathcal{V}}^{\geq 0}$ is a technical problem, which can be solved by a careful examination of the Kuranishi model for $\mathcal{V}_\alpha$ in a point of $\hat{\mathcal{T}}'_{\beta_0^+}(L)$. The final result is

$$\partial\bar{\mathcal{V}}^{\geq 0} = \hat{\mathcal{W}}^{\geq 0}_{\beta_+} \bigcup \left(-\hat{\mathcal{W}}^{\geq 0}_{\beta_-}\right) - \hat{\mathcal{T}}'_{\beta_0^+}(L) \ .$$

Since the cohomology classes u, $\nu(l_i)$ extend to $\hat{\mathcal{W}}$ and $\mathcal{V}$, and their restrictions to the moduli space $\hat{\mathcal{T}}'_{\beta_0^+}(L)$ coincide with the corresponding classes defined in the section above, the theorem follows from the relation

$$[\hat{\mathcal{W}}^{\geq 0}_{\beta_+}]_{\mathcal{O}} - [\hat{\mathcal{W}}^{\geq 0}_{\beta_-}]_{\mathcal{O}} - [\hat{\mathcal{T}}'_{\beta_0^+}(L)]_{\mathcal{O}_1} = 0$$

between the fundamental classes of the three moduli spaces. Here $\mathcal{O}$ is the orientation determined by $\mathcal{O}_1$ and $\mathbf{H}_0$.

■

Remark: Let (X,g) be a manifold of positive scalar curvature, and let $c \in H^2(X,\mathbb{Z})$ be a characteristic element such that $(g,0)$ is c-good. Then $SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})(\cdot) = 0$ for at least one element in $\{\pm\}$. This element is determined by the sign in the inequality $\pm c \cdot [\omega_g] < 0$, and the other value of $SW_{X,(\mathcal{O}_1,\mathbf{H}_0)}(\mathfrak{c})(\cdot)$ is determined by the wall crossing formula.

5 Seiberg-Witten invariants of Kähler surfaces

Let (X, g) be a Kähler surface with Kähler form ω_g, and let $\mathfrak{c}_0$ be the class of the canonical $Spin^c(4)$-structure of determinant $K_X^\vee$ on (X, g). The corresponding spinor bundles are $\Sigma^+ = \Lambda^{00} \oplus \Lambda^{02}$, $\Sigma^- = \Lambda^{01}$ [OT1]. There is a natural bijection between classes of $Spin^c(4)$-structures $\mathfrak{c}$ of Chern class c and isomorphism classes of line bundles M with $2c_1(M) - c_1(K_X) = c$. We denote by $\mathfrak{c}_M$ the class defined by a line bundle M. The spinor bundles of $\mathfrak{c}_M$ are the tensor products $\Sigma^\pm \otimes M$, and the map $\gamma_M : \Lambda^1_X \longrightarrow \mathbb{R}SU(\Sigma^+ \otimes M, \Sigma^- \otimes M)$ given by $\gamma_M(\cdot) = \gamma_0(\cdot) \otimes \mathrm{id}_M$ is a Clifford map representing $\mathfrak{c}_M$.

Let C_0 be the Chern connection in the anti-canonical bundle $K_X^\vee$. We use the variable substitutions $A := C_0 \otimes B^{\otimes 2}$ with $B \in \mathcal{A}(M)$ and $\Psi =: \varphi + \alpha \in A^0(M) \oplus A^{02}(M)$ to rewrite the Seiberg-Witten equations for (A, Ψ) in terms of $(B, \varphi + \alpha) \in \mathcal{A}(M) \times [A^0(M) \oplus A^{02}(M)]$.

Proposition 24 *Let (X, g) be a Kähler surface, and $\beta \in A^{1,1}_{\mathbb{R}}$ a closed real $(1,1)$-form in the class b. Let M be a Hermitian line bundle such that $(2c_1(M) - c_1(K_X) - b) \cdot [\omega_g] < 0$. A pair $(B, \varphi + \alpha) \in \mathcal{A}(M) \times [A^0(M) \oplus A^{02}(M)]$ solves the β-twisted Seiberg-Witten equations $(SW^{\gamma_M}_\beta)$ iff:*

$$\left\{\begin{array}{l} F_B^{20} = F_B^{02} = 0 \\ \alpha = 0 \ , \quad \bar\partial_B(\varphi) = 0 \\ i\Lambda_g F_B + \frac{1}{2}\varphi\bar\varphi + (\frac{s}{2} - \pi\Lambda_g\beta) = 0 \ . \end{array}\right.$$

Proof: The pair $(B, \varphi + \alpha)$ solves $(SW^{\gamma_M}_\beta)$ iff

$$\begin{array}{ll} F_A^{20} & = -\varphi \otimes \bar\alpha \\ F_A^{02} & = \ \alpha \otimes \bar\varphi \\ \bar\partial_B(\varphi) & = \ i\Lambda\partial_B(\alpha) \\ i\Lambda_g(F_A + 2\pi i\beta) & = -\left(\varphi\bar\varphi - *(\alpha \wedge \bar\alpha)\right) . \end{array}$$

Using Witten's transformation $(B, \varphi + \alpha) \longmapsto (B, \varphi - \alpha)$, we find $\varphi \otimes \bar\alpha = \alpha \otimes \bar\varphi = 0$, hence $F_A^{20} = F_A^{02} = 0$, so that φ or α must vanish. Putting $c := 2c_1(M) - c_1(K_X)$ and integrating the last equation over X we get

$$\frac{1}{2\pi}\int_X (|\alpha|^2 - |\varphi|^2)\frac{\omega_g^2}{2} = \int_X (\frac{i}{2\pi}F_A - \beta) \wedge \omega_g = (c - b) \cup [\omega_g] < 0 \ ,$$

hence $\alpha = 0$. ∎

Let $\mathcal{D}ou(m)$ be the Douady space of all effective divisors D on X with $c_1(\mathcal{O}_X(D)) = m$.

Theorem 25 *Let (X, g) be a connected Kähler surface, and let $\mathfrak{c}_M$ be the class of the $Spin^c(4)$-structure associated to a Hermitian line bundle M with $c_1(M) = m$. Let $\beta \in A^{1,1}_{\mathbb{R}}$ be a closed form representing the class b such that $(2m - c_1(K_X) - b) \cup [\omega_g] < 0$ (> 0).*
i) If $c \notin NS(X)$, then $\mathcal{W}^{\gamma_M}_{X,\beta} = \emptyset$. If $c \in NS(X)$, then there is a natural real analytic isomorphism $\mathcal{W}^{\gamma_M}_{X,\beta} \simeq \mathcal{D}ou(m)$ $(\mathcal{D}ou(c_1(K_X) - m))$.
ii) $\mathcal{W}^{\gamma_M}_{X,\beta}$ is smooth at a point corresponding to $D \in \mathcal{D}ou(m)$ iff $h^0(\mathcal{O}_D(D)) = \dim_D \mathcal{D}ou(m)$. This condition is always satisfied when $h^1(\mathcal{O}_X) = 0$.
iii) If $\mathcal{W}^{\gamma_M}_{X,\beta}$ is smooth at a point corresponding to D, then it has the expected dimension in this point iff $h^1(\mathcal{O}_D(D)) = 0$.

Proof: Clearly $c \in NS(X)$ is a necessary condition for $\mathcal{W}^{\gamma_M}_{X,\beta} \neq \emptyset$. Putting again $c := 2m - c_1(K_X)$, we may assume that we are in the case $(c-b)\cdot[\omega_g] < 0$, since the other one can be reduced to it by Serre duality. Under this assumption $\mathcal{W}^{\gamma_M}_{X,\beta}$ can be identified with the moduli space of holomorphic pairs $(\bar\partial, \varphi) \in \mathcal{H}(M) \times A^0(M)$ for which the generalized vortex equation

$$i\Lambda_g F_h + \frac{1}{2}\varphi\bar\varphi^h + (\frac{s}{2} - \pi\Lambda_g\beta) = 0$$

is solvable. The latter space is naturally isomorphic with the Douady space $\mathcal{D}ou(m)$ [OT1]. The remaining assertions follow from the long exact cohomology sequence of the structure sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\cdot D} \mathcal{O}_X(D) \longrightarrow \mathcal{O}_D(D) \longrightarrow 0 \ .$$

∎

Remark: We use the complex structure of the surface to orient $H^1(X, \mathbb{R})$ and $\mathbb{H}^2_{+,g}(X)$. With this convention, the following holds:

The natural isomorphism $\mathcal{W}^{\gamma_M}_{X,\beta} \simeq \mathcal{D}ou(m)$ respects the orientation when $(2m - c_1(K_X) - b) \cup [\omega_g] < 0$. If $(2m - c_1(K_X) - b) \cup [\omega_g] > 0$, then the isomorphism $\mathcal{W}^{\gamma_M}_{X,\beta} \simeq \mathcal{D}ou(c_1(K_X) - m)$ multiplies the orientation by the factor $(-1)^{\chi(M)}$.

The pull-back of the hyperplane class of $\mathcal{D}ou(m)$ is precisely u when $(2m - c_1(K_X) - b) \cup [\omega_g] < 0$. If $(2m - c_1(K_X) - b) \cup [\omega_g] > 0$, then the pull-back of the hyperplane class of $\mathcal{D}ou(c_1(K_X) - m)$ is $-u$.

Recall that an effective divisor D on a connected complex surface X is k-connected iff $D_1 \cdot D_2 \geq k$ for every effective decomposition $D = D_1 + D_2$ [BPV]. Canonical divisors of minimal surfaces of Kodaira dimension $\kappa = 1, 2$ are $(\kappa - 1)$-connected.

Lemma 26 *Every connected complex surface with $p_g > 0$ is oriented diffeomorphic to a surface which possesses a 0-connected canonical divisor.*

Proof: Let X be a connected complex surface with $p_g > 0$ and minimal model $X_{\min}$. Let $K_{\min}$ be a 0-connected canonical divisor of $X_{\min}$. Choose $b_2(X) - b_2(X_{\min})$ distinct points $x_i \in X_{\min} \setminus \mathrm{supp}(K_{\min})$, and note that X is diffeomorphic to the blow up $\hat{X}_{\min}$ of $X_{\min}$ in these points. Denote by σ the projection $\sigma : \hat{X}_{\min} \longrightarrow X_{\min}$ and by E the exceptional divisor. Then $\hat{K} := \sigma^*(K_{\min}) + E$ is a canonical divisor on $\hat{X}_{\min}$. If $\hat{K}$ decomposes as $\hat{K} = D_1 + D_2$, then every component of E is contained in precisely one of the summands, and $K_{\min} = \sigma_*(\hat{K}) = \sigma_* D_1 + \sigma_* D_2$ is a decomposition of $K_{\min}$. This implies $D_1 \cdot D_2 = \sigma_* D_1 \cdot \sigma_* D_2 \geq 0$. ■

Corollary 27 *([W]) All non-trivial Seiberg-Witten invariants of Kähler surfaces with $p_g > 0$ have index 0.*

Proof: Let X be a Kähler surface with $p_g > 0$. We may suppose that X possesses a 0-connected canonical divisor K, defined by a holomorphic 2-form η. Using a moduli space $\mathcal{W}^{\gamma}_{X,\eta}$ to calculate the invariant as in [W], we find an effective decomposition $K = D_1 + D_2$. This implies $w_c = -D_1 \cdot D_2 \leq 0$. ■

Corollary 28 *Let X be a Kählerian surface with $p_g = 0$ and $q = 0$. Endow $H^1(X, \mathbb{R}) = 0$ with the standard orientation and let $\mathbf{H}_0$ be the component of $\mathbf{H}$ containing Kähler forms . If $m(m - c_1(K_X)) \geq 0$, then we have*

$$SW_{X,\mathbf{H}_0}(\mathfrak{c}_M)(+) = \begin{cases} 1 & \text{if} \quad \mathcal{D}ou(m) \neq \emptyset \\ 0 & \text{if} \quad \mathcal{D}ou(m) = \emptyset \ , \end{cases}$$

and

$$SW_{X,\mathbf{H}_0}(\mathfrak{c}_M)(-) = \begin{cases} 0 & \text{if} \quad \mathcal{D}ou(m) \neq \emptyset \\ -1 & \text{if} \quad \mathcal{D}ou(m) = \emptyset \ . \end{cases}$$

Proof: Suppose $\mathcal{D}ou(m) \neq \emptyset$. Since $p_g = 0$ and $q = 0$, we must have $\mathcal{D}ou(c_1(K_X) - m) = \emptyset$, hence $SW_{X,\mathbf{H}_0}(\mathfrak{c}_M)(-) = 0$ by Theorem 10. Now the wall crossing formula implies $SW_{X,\mathbf{H}_0}(\mathfrak{c}_M)(+) = 1$. If $\mathcal{D}ou(m) = \emptyset$, then $SW_{X,\mathbf{H}_0}(\mathfrak{c}_M)(+) = 0$ by Theorem 10, and $SW_{X,\mathbf{H}_0}(\mathfrak{c}_M)(-) = -1$ again by the wall crossing formula. ■

An interesting formulation is obtained under the additional assumption $\mathrm{Tors}_2 H^2(X, \mathbb{Z}) = 0$. Then a class $\mathfrak{c}$ of $Spin^c(4)$-structures is determined by its Chern class c and $\frac{c_1(K_X) \pm c}{2}$ makes sense. If $c^2 \geq c_1(K_X)^2$, then

$$SW_{X,\mathbf{H}_0}(c)(+) = \left\{ \begin{array}{ll} 1 \ \text{ if } & \mathcal{D}ou(\frac{c_1(K_X)+c}{2}) \neq \emptyset \\ \\ 0 \ \text{ if } & \mathcal{D}ou(\frac{c_1(K_X)+c}{2}) = \emptyset \ , \end{array} \right.$$

and

$$SW_{X,\mathbf{H}_0}(c)(-) = \left\{ \begin{array}{ll} -1 \ \text{ if } & \mathcal{D}ou(\frac{c_1(K_X)-c}{2}) \neq \emptyset \\ \\ 0 \ \text{ if } & \mathcal{D}ou(\frac{c_1(K_X)-c}{2}) = \emptyset \ . \end{array} \right.$$

Example: Let $X = \mathbb{P}^2$, let $h \in H^2(\mathbb{P}^2, \mathbb{Z})$ be the class of the ample generator, and let $\mathbf{H}_0$ be the component of $\mathbf{H} = \{\pm h\}$ which contains h. The classes of $Spin^c(4)$-structures are labelled by odd integers c, and the index corresponding to c is $w_c = \frac{1}{4}(c^2 - 9)$. The chambers of type c contained in $\mathbf{H}_0 \times H^2_{\mathrm{DR}}(X)$ are the half-lines $C_{\mathbf{H}_0,\pm} = \{b \in H^2_{\mathrm{DR}}(\mathbb{P}^2) | \ \pm (c - b) \cdot h < 0\}$. Using Theorem 10 to calculate the moduli spaces $\mathcal{W}^{\gamma_{\frac{c-3}{2}}}_{\mathbb{P}^2,\beta}$ we find

$$\mathcal{W}^{\gamma_{\frac{c-3}{2}}}_{\mathbb{P}^2,\beta} \simeq \left\{ \begin{array}{ll} |\mathcal{O}_{\mathbb{P}^2}(\frac{c-3}{2})| & \text{if} \quad [\beta] > c \\ |\mathcal{O}_{\mathbb{P}^2}(\frac{-c-3}{2})| & \text{if} \quad [\beta] < c \ . \end{array} \right.$$

Taking into account the orientation-conventions, we get by direct verification:

$$SW_{\mathbb{P}^2,\mathbf{H}_0}(c)(+) = \left\{ \begin{array}{ll} 1 & \text{if} \quad c \geq 3 \\ 0 & \text{if} \quad c < 3 \ , \end{array} \right. \quad SW_{\mathbb{P}^2,\mathbf{H}_0}(c)(-) = \left\{ \begin{array}{ll} -1 & \text{if} \quad c \leq -3 \\ 0 & \text{if} \quad c > -3 \ . \end{array} \right.$$

For every c, the subspace $\mathbf{H}_0 \times \{0\}$ is contained in the chamber on which $SW_{\mathbb{P}^2,\mathbf{H}_0}(c)$ vanishes.

Remark: Let $X = \hat{\mathbb{P}}^2$ be the blow-up of $\mathbb{P}^2$ in $r \geq 3$ points, and fix a non-negative even integer w. There exist infinitely many solutions $(d; m_1, \ldots, m_r) \in \mathbb{N}^{\oplus(r+1)}$ of the equation $\frac{1}{2}w = \frac{1}{2}d(d+3) - \sum\limits_{i=1}^{r} \frac{m_i(m_i+1)}{2}$. For every solution $(d; m_1, \ldots, m_r)$ let M be the underlying line bundle of the linear system $|dL - \sum\limits_{i=1}^{r} m_i E_i|$, and set $c := 2c_1(M) - c_1(K_X)$. Then $w_c = w$ and we have $SW_{X,\mathbf{H}_0}(c)(+) = 1$, $SW_{X,\mathbf{H}_0}(c)(-) = 0$ for the component $\mathbf{H}_0$ containing Kähler classes. Hence there exist infinitely many characteristic classes c with non-trivial Seiberg-Witten invariants and prescribed index $w_c = w$.

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